О некоторых свойствах пространств Гельдера

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В настоящей работе мы приводим некоторые свойства обычных и весовых пространств Гельдера. Несмотря на то, что эти свойства довольно просты, они весьма полезны в различных приложениях. В частности, они весьма полезны в области дифференциальных уравнений с частными производными при получении различных оценок решений в пространствах Гельдера. Мы рассматриваем вопросы эквивалентных нормировок, вложений весовых пространств в обычные невесовые и условия компактности множеств.

Ключевые слова: свойство Гельдера, весовое пространство Гельдера, эквивалентные полунормы, компактность в пространствах Гельдера.

Introduction

In the present paper we show some properties of usual and weighted Hölder spaces. Though these properties are pretty simple, they are very useful in applications. In particular, they are useful in the field of partial differential equations at obtaining different estimates of solutions in Hölder spaces. Some of the properties under consideration were proved earlier in [1, P.4-21] and [2, P.67-79] and we present them here for completeness without the proofs. The corresponding proofs can be found in [1, P.4-21] or [2, P.67-79].

Let us turn now to exact definitions and to the main results. Denote $H=\{x=(x',x_N)\in R^N: x_N>0\}$, $Q=\{(x,t): x\in H, -\infty<t<\infty\}$. And we note at once that all the reasoning and statement below are valid in evident way also for $Q^+=\{(x,t): x\in H, t\geq 0\}$ instead of Q. Let m be a positive integer and let n be a positive number, $n<m$. Denote

$$\omega=n/m<1.$$

Let $C^{\gamma}_{\omega\gamma}(\overline{H})$, $\gamma\in(0,1)$, be the weighted Hölder space of continuous functions $u(x)$ with the finite norm

$$|u|^{(\gamma)}_{\omega\gamma,\overline{H}}\equiv\|u\|_{C^{\gamma}_{\omega}(\overline{H})}\equiv|u|^{(0)}_{\overline{H}}+\langle u\rangle^{(\gamma)}_{\omega\gamma,\overline{H}}, \qquad (1)$$

where

$$|u|^{(0)}_{\overline{H}}=\max_{x\in\overline{H}}|u(x)|, \langle u\rangle^{(\gamma)}_{\omega\gamma,\overline{H}}=\sup_{x,\bar{x}\in\overline{H}}\left(x_N^*\right)^{\omega\gamma}\frac{|u(x)-u(\bar{x})|}{|x-\bar{x}|^{\gamma}}, x_N^*=\max\{x_N,\bar{x}_N\}. \qquad (2)$$

Thus $\langle u\rangle^{(\gamma)}_{\omega\gamma,\overline{H}}$ represents a weighted Hölder constant of the function $u(x)$. We suppose that $n<m$, and if n is a noninteger,

$$(1-\omega)\gamma=\gamma\left(1-\frac{n}{m}\right)<\min(\{n\},1-\{n\}), \qquad (3)$$

where for a real number a, $\{a\}$ is the fractional part of a, $[a]$ is the integer part of a. This assumption is technical and it allows us, for example, to consider the functions x_N^{n-j} as elements of $C^{\gamma}_{\omega\gamma}(\overline{H})$ for all integer $j<n$.

In the similar way we define the Hölder seminorms with respect to each variable separately

$$\langle u\rangle^{(\gamma)}_{\omega\gamma,x_i,\overline{H}}=\sup_{x,\bar{x}\in\overline{H}}\left(x_N^*\right)^{\omega\gamma}\frac{|u(x)-u(\bar{x})|}{h^{\gamma}}, x_N^*=\max\{x_N,\bar{x}_N\}, i=\overline{1,N}, \qquad (4)$$

where $x=(x_1,\dots x_i,\dots,x_N)$, $\bar{x}=(x_1,\dots x_i+h,\dots,x_N)$, $h>0$.

In the standard way we denote by $\langle u\rangle^{(\gamma)}_{x_i,\overline{H}}$, $\langle u\rangle^{(\gamma)}_{x',\overline{H}}$, and $\langle u\rangle^{(\gamma)}_{x,\overline{H}}$ usual unweighted Hölder seminorms with respect to each variable separately, with respect to $x'=(x_1,\dots,x_{N-1})$ or with respect to all x-variables.

Let M be a positive integer. In the space R^M we use standard Hölder spaces $C^{\bar{l}}(R^M)$, where $\bar{l}=(l_1,l_2,\dots,l_M)$, l_i are arbitrary positive non-integers. The norm in such spaces is defined by

$$\|u\|_{C^{\bar{l}}(R^M)}\equiv|u|^{[\bar{l}]}_{R^M}=|u|^{(0)}_{R^M}+\sum_{i=1}^{M}\langle u\rangle^{(l_i)}_{x_i,R^M}, \qquad (5)$$

$$\langle u\rangle^{(l_i)}_{x_i,R^M}=\sup_{x\in R^M,h>0}\frac{\left|D^{[l_i]}_{x_i}u(x_1,\dots,x_i+h,\dots,x_M)-D^{[l_i]}_{x_i}u(x)\right|}{h^{l_i-[l_i]}}, \qquad (6)$$

where $[l_i]$ is the integer part of the number l_i, $D^{[l_i]}_{x_i}u$ is the derivative of order $[l_i]$ with respect to the variable x_i of a function u.

Proposition 1.

Seminorm (6) can be equivalently defined by ([3, P.361-272], [4, P.215-221], [5, P.261-273])

$$\langle u\rangle^{(l_i)}_{x_i,R^M}\simeq\sup_{x\in R^M,h>0}\frac{\left|\Delta^k_{h,x_i}u(x)\right|}{h^{l_i}}, k>l_i, \qquad (7)$$

where $\Delta_{h,x_i}u(x)=u(x_1,\dots,x_i+h,\dots,x_N)-u(x)$ *is the difference from a function* $u(x)$ *with respect to the variable* x_i *with a step* h, $\Delta^k_{h,x_i}u(x)=\Delta_{h,x_i}\left(\Delta^{k-1}_{h,x_i}u(x)\right)=\left(\Delta_{h,x_i}\right)^k u(x)$ *is the difference of power* k.

The same is also valid not only for the whole space R^M but also for it's subsets of the form $R^M\cap\{x_{i_1},x_{i_2},\dots,x_{i_K}\geq 0\}$ with $K\leq M$. Note that below we prove an analogous statement for weighted spaces.

It is known that functions from the space $C^{\bar{l}}(R^M)$ have also mixed derivatives up to definite orders and all derivatives are Hölder continuous with respect to all variables with some exponents in accordance with ratios between the exponents l_i. Namely, if $\bar{k}=(k_1,\dots,k_M)$ with nonnegative integers k_i, $k_i\leq[l_i]$, and

$$\omega=1-\sum_{i=1}^{N}\frac{k_i}{l_i}>0, \qquad (8)$$

then (see, for example, [4])

$$D_x^{\bar{k}}u(x)\in C^{\bar{d}}(R^M),\ \|D_x^{\bar{k}}u\|_{C^{\bar{d}}(R^M)}\le C\|u\|_{C^{|\bar{d}|}(R^M)},\qquad(9)$$

where

$$\bar{d}=(d_1,\dots,d_M),\ d_i=\omega l_i.\qquad(10)$$

Moreover, relation (9) is valid not only for R^M but for any domain $\Omega\subset R^M$ with sufficiently smooth boundary and we have

$$\|D_x^{\bar{k}}u\|_{C^{\bar{d}}(\bar{\Omega})}\le C\|u\|_{C^{l}(\bar{\Omega})}.\qquad(11)$$

For special domains of the form $\Omega_+ = R^M\cap\{x_{i_1},x_{i_2},\dots,x_{i_k}\ge 0\}$ we have even more strong inequality just for seminorms

$$\sum_{\bar{k}}\sum_{i=1}^{M}\left\langle D_x^{\bar{k}}u\right\rangle_{x_i,\bar{\Omega}_+}^{(d_i)}\le C\sum_{i=1}^{M}\langle u\rangle_{x_i,\bar{\Omega}_+}^{(l_i)}.\qquad(12)$$

Here the sum is taken over all $\bar{k}$ with the property (8) and d_i are defined in (10). The analog of this estimate for an arbitrary smooth domain Ω (including bounded domains) is

$$\sum_{\bar{k}}\sum_{i=1}^{M}\left\langle D_x^{\bar{k}}u\right\rangle_{x_i,\bar{\Omega}}^{(\hat{d}_i)}\le C\left(\sum_{i=1}^{M}\langle u\rangle_{x_i,\bar{\Omega}}^{(l_i)}+|u|_{\bar{\Omega}}^{(0)}\right)\qquad(13)$$

with arbitrary $\hat{d}_i\le d_i$.

Embedding and equivalent seminorms

It turns out that the weighted space $C_{\omega\gamma}^{\gamma}(\bar{H})$ is embedded into the usual space $C^{\gamma-\omega\gamma}(\bar{H})$. Namely, we have the following assertion, which we present here without a proof. The complete proof can be found in , .

Proposition 2. ***Let a function*** $u(x)\in C_{\omega\gamma}^{\gamma}(\bar{H})$. ***Then*** $u(x)$ ***is continuous in*** $\bar{H}$ ***and***

$$\langle u\rangle_{x,H}^{(\gamma-\omega\gamma)}\le C\langle u\rangle_{\omega\gamma,x,\bar{H}}^{(\gamma)}.\qquad(14)$$

We need also the analog of relation (7) for weighted seminorm.

Proposition 3. ***Let*** $l=m+\gamma>0$ ***be noninteger,*** $m=[l]$, , $\gamma\in(0,1)$, ***and let a function*** $u(y)\in C_{\omega\gamma}^{l}([0,\infty))$, $\omega\in(0,1)$, ***in the sense that***

$$\left\langle D_y^m u\right\rangle_{\omega\gamma,y}^{(\gamma)}=\sup_{y,h>0}(y+h)^{\omega\gamma}\frac{\left|D_y^m u(y+h)-D_y^m u(y)\right|}{h^{\gamma}}<\infty.\qquad(15)$$

Then for any integer $k>l$

$$\left\langle D_y^m u\right\rangle_{\omega\gamma,y}^{(\gamma)}\le C_k\sup_{y,h>0}y^{\omega\gamma}\frac{\left|\Delta_h^k u(y)\right|}{h^{l}}\equiv C_k\langle u\rangle_{\omega\gamma,y}^{(l|k)},\qquad(16)$$

where $\Delta_h^k u(y)$ *is the* k*-th difference with the step* h, $\Delta_h^1 u(y)=\Delta_h u(y)=u(y+h)-u(y)$, $\Delta_h^k u(y)=\Delta_h\left(\Delta_h^{k-1}u(y)\right)$. *Note that the inverse inequality to (16) is evident because of the mean value theorem.*

For the proof of this proposition see [1, P.4-21], [2, P.67-79]. As a corollary we have the following assertion.

Proposition 4.

Let a function $u(x)$ *be defined in* $\bar{H}$ *and*

$$\langle u\rangle_{\omega\gamma,x,\bar{H}}^{(\gamma)}\equiv\sup_{x,\bar{h}\in\bar{H}}(x_N+h_N)^{\omega\gamma}\frac{\left|u(x+\bar{h})-u(x)\right|}{|\bar{h}|^{\gamma}}<\infty,\ \gamma\in(0,1).$$

Then for any integer $k\ge 1$ *there is a constant* $C_k^{(i)}=C^{(i)}(k,N,\gamma,\omega)$, $i=1,2$ *with*

$$\langle u\rangle_{\omega\gamma,x,\bar{H}}^{(\gamma)}\le C_k^{(1)}\sup_{x,\bar{h}\in\bar{H}}x_N^{\ \omega\gamma}\frac{\left|\Delta_{\bar{h}}^k u(x)\right|}{|\bar{h}|^{\gamma}}\equiv C_k^{(1)}\langle\langle u\rangle\rangle_{\omega\gamma,x,\bar{H}}^{(\gamma|k)}\qquad(17)$$

and

$$\langle\langle u\rangle\rangle_{\omega\gamma,x,\bar{H}}^{(\gamma|k)}\le C_k^{(2)}\langle u\rangle_{\omega\gamma,x,\bar{H}}^{(\gamma)}.\qquad(18)$$

The proof of this Proposition can be found in [1, P.4-21], [2, P.67-79].

Corollary 5. ***The seminorms***

$$\langle u\rangle_{\omega\gamma,\bar{H}}^{(\gamma)}=\sup_{x,\bar{x}\in\bar{H}}\left(x_N^{*}\right)^{\omega\gamma}\frac{\left|u(x)-u(\bar{x})\right|}{|x-\bar{x}|^{\gamma}},\ x_N^{*}=max\{x_N,\bar{x}_N\}$$

with $x_N^{*}=max\{x_N,\bar{x}_N\}$ *and*

$$\widehat{\langle u\rangle}_{\omega\gamma,\bar{H}}^{(\gamma)}=\sup_{x,\bar{x}\in\bar{H}}\left(\hat{x}_N\right)^{\omega\gamma}\frac{\left|u(x)-u(\bar{x})\right|}{|x-\bar{x}|^{\gamma}},\ \hat{x}_N=min\{x_N,\bar{x}_N\}$$

with $\hat{x}_N=min\{x_N,\bar{x}_N\}$ *are equivalent.*

For the proof of this corollary it is enough to choose $k=1$ in (17).

Sufficient condition and compactness

We prove in addition two useful lemmas about Hölder spaces. First we prove some lemma that makes the verification of the Hölder condition for functions on domains with boundaries simpler. This is done by restricting the general position of two different points of a domain to the situation when the two points are away from a boundary in some sense.

Lemma 6.

Let $\gamma\in(0,1)$ *and* $\omega\in[0,1)$ *(the case* $\omega=0$ *corresponds to the nonweighted case). Let a function* $u(y)$, $y\in R^{+}\equiv(0,\infty)$ *satisfy the condition*

$$\sup_{0<h\le\varepsilon y}y^{\omega\gamma}\frac{\left|u(y+h)-u(y)\right|}{h^{\gamma}}\equiv\langle u\rangle_{\omega\gamma,R^{*}}^{(\gamma|\varepsilon-)}<\infty.\qquad(19)$$

Then $u(y)$ *is continuous on* $[0,\infty)$ *and*

$$\langle u\rangle_{\omega\gamma,\bar{R}^{*}}^{(\gamma)}\le C_\gamma\varepsilon^{-1}\langle u\rangle_{\omega\gamma,R^{*}}^{(\gamma|\varepsilon-)}.\qquad(20)$$

Proof. Due to Corollary 5 and (19) it is enough to verify that

$$\sup_{\varepsilon y\le h}y^{\omega\gamma}\frac{\left|u(y+h)-u(y)\right|}{h^{\gamma}}\le C_\gamma\varepsilon^{-1}\langle u\rangle_{\omega\gamma,R^{*}}^{(\gamma|\varepsilon-)}.\qquad(21)$$

Let $y,h\ge\varepsilon y>0$ be arbitrary, $\Delta_h u(y)\equiv u(y+h)-u(y)$. Denote $([a]$ is the integer part of $a)$

$$M=\left[\log_{(1+\varepsilon)}\left(\frac{y+h}{y}\right)\right]\quad if\quad\log_{(1+\varepsilon)}\left(\frac{y+h}{y}\right)\text{ is a noninteger or}$$

$$M=\log_{(1+\varepsilon)}\left(\frac{y+h}{y}\right)-1\quad if\quad\log_{(1+\varepsilon)}\left(\frac{y+h}{y}\right)\ is\ an\ integer.$$

Consider the difference

$$u(y+h)-u(y)=\Delta_h u(y)=\sum_{i=1}^{M}\left(u(y_{i+1})-u(y_i)\right),$$

where

$$y_1=y,\ y_i=y_{i-1}+\varepsilon y_{i-1}=(1+\varepsilon)y_{i-1}=(1+\varepsilon)^{i-1}y,\ i\le M,\ y_{M+1}=y+h,$$

so that $(y_{i+1}-y_i)=\varepsilon y_i$. We have

$$y^{\omega\gamma}\frac{\left|u(y+h)-u(y)\right|}{h^{\gamma}}\le\sum_{i=1}^{M}y_i^{\omega\gamma}\frac{\left|u(y_{i+1})-u(y_i)\right|}{\left|y_{i+1}-y_i\right|^{\gamma}}\left(\frac{\left|y_{i+1}-y_i\right|}{h^{\gamma}}\right)^{\gamma}\le$$

$$\le\langle u\rangle_{\omega\gamma,R^{*}}^{(\gamma|\varepsilon-)}\sum_{i=1}^{M}\left(\frac{\left|y_{i+1}-y_i\right|}{h}\right)^{\gamma}\equiv\langle u\rangle_{\omega\gamma,R^{*}}^{(\gamma|\varepsilon-)}S.$$

On the other hand

$$S\equiv\sum_{i=1}^{M}\left(\frac{\left|y_{i+1}-y_i\right|}{h}\right)^{\gamma}\le\sum_{i=1}^{M}\left(\frac{\varepsilon(1+\varepsilon)^{i-1}y}{h}\right)^{\gamma}=$$

$$=\varepsilon^{\gamma}\left(\frac{y}{h}\right)^{\gamma}\sum_{i=1}^{M}(1+\varepsilon)^{\gamma(i-1)}=\varepsilon^{\gamma}\left(\frac{y}{h}\right)^{\gamma}\frac{(1+\varepsilon)^{\gamma M}-1}{(1+\varepsilon)^{\gamma}-1}.$$

But according to the definition of the number M

$$(1+\varepsilon)^{\gamma M}\le\left(\frac{y+h}{y}\right)^{\gamma},$$

so that

$$S=\left(\frac{\varepsilon^{\gamma}}{(1+\varepsilon)^{\gamma}-1}\right)\left[\left(\frac{y}{h}\right)^{\gamma}\left[(1+\varepsilon)^{\gamma M}-1\right]\right]\le C_\gamma\varepsilon^{-1+\gamma}\left(\frac{y+h}{h}\right)^{\gamma}\le C_\gamma\varepsilon^{-1}.$$

From this (21) follows for $y>0$. That is on the open set $(0,\infty)$

$$\langle u\rangle_{\omega\gamma,(0,\infty)}^{(\gamma)}\le C_\gamma\varepsilon^{-1}\langle u\rangle_{\omega\gamma,R^{*}}^{(\gamma|\varepsilon-)}.$$

Then from Corollary 5 and the proof of Proposition 2 it follows that

$$\langle u\rangle_{0,\infty)}^{(\gamma-\omega\gamma)}\le C_\gamma\varepsilon^{-1}\langle u\rangle_{\omega\gamma,R^{*}}^{(\gamma|\varepsilon-)}.$$

This means that $u(y)$ has a finite limit as $y\to 0$ and consequently can be defined at $y=0$ as a continuous function with (20). Thus the lemma follows.

Corollary 7.

Let $\gamma\in(0,1]$ *and* $\omega\in[0,1)$ *(the case* $\omega=0$ *corresponds to the nonweighted case). Let a function* $u(x)$, $x\in H$ *satisfy the condition*

$$\sup_{\bar{h}\in H,|\bar{h}|\le\varepsilon x_N}x_N^{\ \omega\gamma}\frac{\left|u(x+\bar{h})-u(x)\right|}{|\bar{h}|^{\gamma}}\equiv\langle u\rangle_{\omega\gamma,H}^{(\gamma|\varepsilon-)}<\infty.\qquad(22)$$

Then $u(x)$ *is continuous on* $\bar{H}$ *and*

$$\langle u\rangle_{\omega\gamma,\bar{H}}^{(\gamma)}\le C_\gamma\varepsilon^{-1-\gamma}\langle u\rangle_{\omega\gamma,H}^{(\gamma|\varepsilon-)}.\qquad(23)$$

Proof. In view of Lemma 6 for arbitrary fixed x' we have

$$\langle u(x',\cdot)\rangle_{\omega\gamma,x_N,[0,\infty)}^{(\gamma)}\le C_\gamma\varepsilon^{-1}\langle u\rangle_{\omega\gamma,H}^{(\gamma|\varepsilon-)}.\qquad(24)$$

Therefore, it is enough to consider the Hölder property of $u(x)$ with respect to the tangent variables x' only under the condition $|\bar{h}|\ge\varepsilon x_N$. That is for a given $\bar{h}'=(h_1,\dots,h_{N-1})$, and for $x_N>0$ with $|\bar{h}'|\ge\varepsilon x_N$ we must estimate the expression

$$A(x,\bar{h}')\equiv x_N^{\ \omega\gamma}\frac{\left|u(x'+\bar{h}',x_N)-u(x',x_N)\right|}{|\bar{h}'|^{\gamma}}.$$

We estimate $A(x,\bar{h}')$ as follows

$$A(x,\bar{h}')\le x_N^{\ \omega\gamma}\frac{\left|u(x'+\bar{h}',x_N+\varepsilon^{-1}|\bar{h}'|)-u(x',x_N+\varepsilon^{-1}|\bar{h}'|)\right|}{|\bar{h}'|^{\gamma}}+$$

$$+x_N^{\ \omega\gamma}\frac{\left|u(x'+\bar{h}',x_N+\varepsilon^{-1}|\bar{h}'|)-u(x'+\bar{h}',x_N)\right|}{|\bar{h}'|^{\gamma}}+$$

$$+x_N^{\ \omega\gamma}\frac{\left|u(x',x_N+\varepsilon^{-1}|\bar{h}'|)-u(x',x_N)\right|}{|\bar{h}'|^{\gamma}}.$$

In view of (22) and (24) the proof of the corollary is completed now exactly as in Lemma 6.

Now we prove a simple assertion about compactness and convergence in weighted Hölder spaces. This assertion is "almost well known". But the experience of the author shows that the following simple fact is not generally known: after convergence of a sequence from a Hölder space in a weaker Hölder space the limit belongs to the original space. This fact is a very useful tool in applications because smooth functions are not dense in Hölder spaces. The precise statement is as follows.

Proposition 8.

Let $\gamma \in (0,1)$, $\omega \in [0,1)$. *Let* $K \subset \overline{H}$ *be a compact domain in* $\overline{H}$ *with smooth boundary. Let* $U \subset C^{\gamma}_{\omega\gamma}(K)$ *be a bounded subset in* $C^{\gamma}_{\omega\gamma}(K)$ *that is*

$$u(x) \in U \Rightarrow \|u\|_{C^{\gamma}_{\omega\gamma}(K)} \le M \quad (25)$$

for some fixed constant $M > 0$.

Then there exists a sequence $\{u_n(x)\} \subset U$ *and a function* $u_0(x) \in C^{\gamma}_{\omega\gamma}(K)$ *from the same space* $C^{\gamma}_{\omega\gamma}(K)$ *such that for any* $\gamma' \in (0,\gamma)$

$$\|u_n - u_0\|_{C^{\gamma'}_{\omega\gamma'}(K)} + \|u_n - u_0\|_{C^{(1-\omega)\gamma'}(K)} \to_{n\to\infty} 0, \|u_0\|_{C^{\gamma}_{\omega\gamma}(K)} \le M. \quad (26)$$

Proof. From Proposition 2 and from (25) it follows that

$$u(x) \in U \Rightarrow \|u\|_{C^{(1-\omega)\gamma}(K)} \le CM.$$

Thus, as it is well known, there exists a sequence $\{u_n(x)\} \subset U$ and a function $u_0(x) \in \cap_{\gamma' \in (0,\gamma)} C^{(1-\omega)\gamma'}(K)$ with

$$\|u_n - u_0\|_{C^{(1-\omega)\gamma'}(K)} \to_{n\to\infty} 0, \gamma' \in (0,\gamma). \quad (27)$$

Let us show that $u_0(x)$ belongs to the original space $C^{\gamma}_{\omega\gamma}(K)$ and the estimate in (26) is valid. Let $x \in \overline{K}$ and $\overline{h} \ne 0 \in H$ be fixed and such that $x + \overline{h} \in \overline{K}$. Consider the expression

$$A_n(x,\overline{h}) = x_N^{\omega\gamma} \frac{|u_n(x+\overline{h}) - u_n(x)|}{|\overline{h}|^{\gamma}} \le M \quad (28)$$

and suppose that $x_N > 0$ because in the case $x_N = 0$ we have $A_n(x,\overline{h}) = 0$. From (27) it follows that $u_n(x) \to u_0(x)$ uniformly on K. Therefore letting $n \to \infty$ in (28), we obtain

$$A_0(x,\overline{h}) = x_N^{\omega\gamma} \frac{|u_0(x+\overline{h}) - u_0(x)|}{|\overline{h}|^{\gamma}} \le M.$$

Since x and $\overline{h}$ are arbitrary we infer from the last inequality and from (27)

$$u_0(x) \in C^{\gamma}_{\omega\gamma}(K), \|u_0\|_{C^{\gamma}_{\omega\gamma}(K)} \le M.$$

Let us show now that

$$\|u_n - u_0\|_{C^{\gamma'}_{\omega\gamma'}(K)} \to_{n\to\infty} 0, \gamma' \in (0,\gamma). \quad (29)$$

Let $x \in \overline{K}$ and $\overline{h} \ne 0 \in H$ be such that $x + \overline{h} \in \overline{K}$ and let $\gamma' \in (0,\gamma)$. Denote $v_n(x) = u_n(x) - u_0(x)$. and consider the expression

$$A_n(x,\overline{h}) = x_N^{\omega\gamma'} \frac{|v_n(x+\overline{h}) - v_n(x)|}{|\overline{h}|^{\gamma'}}.$$

Let we are given an $\varepsilon > 0$. If $x_N = 0$ then $A_n(x,\overline{h}) = 0$ therefore we suppose that $x_N > 0$. Denote $R_K = inf\{R > 0 : K \subset \{0 \le x_N \le R\}\}$ and consider two cases. If $|\overline{h}| \le \varepsilon x_N$ then we have

$$A_n(x,\overline{h}) = x_N^{\omega\gamma'} |\overline{h}|^{\gamma-\gamma'} \frac{|v_n(x+\overline{h}) - v_n(x)|}{|\overline{h}|^{\gamma}} \le$$

$$\le \varepsilon^{\gamma-\gamma'} R_K^{(1-\omega)(\gamma-\gamma')} \left(x_N^{\omega\gamma} \frac{|v_n(x+\overline{h}) - v_n(x)|}{|\overline{h}|^{\gamma}} \right) \le$$

$$\le \varepsilon^{\gamma-\gamma'} R_K^{(1-\omega)(\gamma-\gamma')} \left(\langle u_n \rangle^{(\gamma)}_{\omega\gamma,K} + \langle u_0 \rangle^{(\gamma)}_{\omega\gamma,K} \right) \le$$

$$\le \varepsilon^{\gamma-\gamma'} 2M R_K^{(1-\omega)(\gamma-\gamma')}. \quad (30)$$

If now $|\overline{h}| > \varepsilon x_N$ then

$$A_n(x,\overline{h}) = \left(\frac{x_N}{|\overline{h}|} \right)^{\omega\gamma'} \frac{|v_n(x+\overline{h}) - v_n(x)|}{|\overline{h}|^{(1-\omega)\gamma'}} \le \varepsilon^{-\omega\gamma'} \langle u_n - u_0 \rangle^{((1-\omega)\gamma')}_{K}. \quad (31)$$

Since x and $\overline{h}$ are arbitrary, from (30) and (31) it follows that

$$\langle u_n - u_0 \rangle^{(\gamma')}_{\omega\gamma',K} \le \varepsilon^{\gamma-\gamma'} C(M,K) + \varepsilon^{-\omega\gamma'} \langle u_n - u_0 \rangle^{((1-\omega)\gamma')}_{K}.$$

Taking into account (27), we have

$$\|u_n - u_0\|_{C^{\gamma'}_{\omega\gamma'}(K)} \le \varepsilon^{\gamma-\gamma'} C(M,K) + \varepsilon^{-\omega\gamma'} \|u_n - u_0\|_{C^{(1-\omega)\gamma'}(K)}.$$

From this we see that the left hand side can be made arbitrary small for large n by choosing first ε sufficiently small and then $n \ge N(\varepsilon)$ sufficiently large.

This completes the proof of the proposition.

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On some properties of Hölder spaces

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In the present paper we prove some properties of usual and weighted Hölder spaces. Though these properties are pretty simple, they are very useful in applications. In particular, they are useful in the field of partial differential equations at obtaining different estimates of solutions in Hölder spaces. We consider such questions as equivalent seminorms, embedding of weighted spaces in unweighted and conditions for compactness.

Keywords: Hölder property, weighted Hölder spaces, equivalent seminorms, compactness in Hölder spaces.

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