О функциональных пространствах с частичным свойством Гельдера

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Мы рассматриваем два типа пространств со свойством Гельдера относительно только части независимых переменных. Наряду с полунормой Гельдера, один тип включает равномерную норму, а другой включает интегральную норму вместо равномерной. Как это было показано ранее, когда свойство Гельдера имеет место для всех независимых переменных, тогда равномерная норма может быть оценена через интегральную норму и полунорму Гельдера. Таким образом, в случае ограниченной области определения эти два типа пространств совпадают. Однако, в случае частичного свойства Гельдера равномерная норма не может, вообще говоря, быть оценена указанным способом, следовательно, соответствующие функции могут быть неограничены. Для обоих этих типов пространств мы доказываем, что они являются банаховыми, то есть полными нормированными пространствами.
Ключевые слова: свойство Гельдера, частичное свойство Гельдера, функциональное пространство, интегральная норма, банахово пространство

Introduction

The present paper deals with functional spaces with partial Hölder property that is with the spaces of functions with the Hölder property with respect to only a part of their variables. Along with different anisotropic Hölder spaces, these spaces are a useful tool in investigating various problems for partial differential equations - see, for example, [1, P.325-341], [2, P.2001-2009] , [3, P.467-471], [4, P.2793-2801], [5, P.393-422] and the references therein. (In particular, such spaces arise naturally when we consider semigroups of operators with parameter $t>0$ and with a generator defined on some Hölder space.)

Turn now to the rigorous definition of the spaces under consideration. Let for a positive integer $M>0$

$$\beta=(\beta_1,...,\beta_M),\beta_k\in(0,1),k=\overline{1,M}.(1.1)$$

We denote by $C^{\beta}(R^M)$ the anisotropic space of continuous in R^M functions $u(x)$ with the finite norm

$$\|u\|_{C^{\beta}(R^M)}\equiv|u|^{(\beta)}_{R^M}\equiv|u|^{(0)}_{R^M}+\sum_{k=1}^{M}\langle u\rangle^{(\beta_k)}_{x_k,R^M},\qquad(1.2)$$

where

$$|u|^{(0)}_{R^M}\equiv\sup_{x\in R^M}|u(x)|,(1.3)$$

$$\langle u\rangle^{(\beta_k)}_{x_k,R^M}\equiv\sup_{x\in R^M,h>0}\frac{|u(x_1,...,x_k+h,...,x_M)-u(x)|}{h^{\beta_k}}.\qquad(1.4)$$

Besides the space $C^{\beta}(R^M)$ denote also $H^{\beta}_p(R^M)\equiv C^{\beta}(R^M)\cap L_p(R^M)$ with the norm

$$\|u\|_{H^{\beta}_p(R^M)}=\|u\|_{L_p(R^M)}+\sum_{k=1}^{M}\langle u\rangle^{(\beta_k)}_{x_k,R^M},p>1.(1.5)$$

We omit the term $|u|^{(0)}_{R^M}$ in (1.5) because it can be proved (see [3, P.467-471], [4, P.2793-2801] and also see [5, formula (1.9)] and [6, formula (1.20)]) that for $u(x)\in H^{\beta}_p(R^M)$

$$|u|^{(0)}_{R^M}\le C_{M,\beta,p}\|u\|_{H^{\beta}_p(R^M)}(1.6)$$

with some constant $C_{M,\beta,p}$ depending only on M , p, and β.

Let now N_1 and N_2 be positive integers, $N=N_1+N_2$, $R^N=R^{N_1}\times R^{N_2}$,

$$x\in R^N,x=(x^{(1)},x^{(2)}),x^{(1)}\in R^{N_1},x^{(2)}\in R^{N_2}(1.7)$$

and we denote $x^{(1)}=(x_1,...,x_{N_1})$, $x^{(2)}=(x_{N_1+1},...,x_{N_1+N_2})$. Let also

$$\alpha=(\alpha_1,...,\alpha_{N_1}),\alpha_k\in(0,1),k=\overline{1,N_1}.(1.8)$$

Denote by $C^{\alpha,0}_{x^{(1)}}(R^N)$ the space of bounded functions with the Hölder property with respect to the variables $x^{(1)}$ only with the norm

$$\|u\|_{C^{\alpha,0}_{x^{(1)}}(R^N)}\equiv|u|^{(\alpha,0)}_{x^{(1)},R^N}\equiv|u|^{(0)}_{R^N}+\sum_{k=1}^{N_1}\langle u\rangle^{(\alpha_k)}_{x_k,R^N}.(1.9)$$

At last, analogously (1.5) and (1.9), define the space $H^{\alpha,0}_{x^{(1)},p}(R^N)$ of functions $u(x)$ with the finite norm

$$\|u\|_{H^{\alpha,0}_{x^{(1)},p}(R^N)}=\|u\|_{L_p(R^N)}+\sum_{k=1}^{N_1}\langle u\rangle^{(\alpha_k)}_{x_k,R^N},p>1.(1.10)$$

And it is important to note here that the functions from this space are not necessarily bounded in R^N.

Now we can formulate the main assertion of the present paper.

Theorem 1. ***The spaces $C^{\alpha,0}_{x^{(1)}}(R^N)$ and $H^{\alpha,0}_{x^{(1)},p}(R^N)$ are Banach spaces that is they are full normed spaces.***

The proof of this Theorem will be given in the following sections.

The proof of the space $C^{\alpha,0}_{x^{(1)}}(R^N)$ is a Banach space

We start with more simple case of the space $C^{\alpha,0}_{x^{(1)}}(R^N)$. Evidently, it is enough to prove that $C^{\alpha,0}_{x^{(1)}}(R^N)$ is full. So let us assume we are given a sequence $\{u_n(x);n=1,2,...\}\subset C^{\alpha,0}_{x^{(1)}}(R^N)$ with the properties

$$\|u_n\|_{C^{\alpha,0}_{x^{(1)}}(R^N)}\equiv|u_n|^{(\alpha,0)}_{x^{(1)},R^N}\le C_1,n=1,2,..,(2.1)$$

$$\|u_n-u_m\|_{C^{\alpha,0}_{x^{(1)}}(R^N)}\to0,n,m\to\infty,n,m=1,2,...(2.2)$$

For any arbitrary but fixed point $y\in R^N$ we have from (2.2)

$$|u_n(y)-u_m(y)|\to0,n,m\to\infty$$

that is the numerical sequence $\{u_n(y)\}$ is fundamental and hence it has its limit as $n\to\infty$. Denote this limit for each fixed y by $u(y)$ which defines corresponding function $u(x)$ on whole R^N. Fix now arbitrary $\varepsilon>0$. Then from (2.2) it follows that there exists a number $M_0>0$ such that

$$\sup_{x\in R^N}|u_n(x)-u_m(x)|\le\varepsilon,n,m>M_0.(2.3)$$

Since for each $x\in R^N$ we have $u_m(x)\to u(x)$ as $m\to\infty$, then going in (2.3) to the limit as $m\to\infty$ for each $x\in R^N$, we obtain

$$\sup_{x\in R^N}|u_n(x)-u(x)|\le\varepsilon,n>M_0,$$

that is, taking into account also (2.1),

$$|u_n-u|^{(0)}_{R^N}\le\varepsilon,n>M_0,|u|^{(0)}_{R^N}\le C_1,(2.4)$$

or in other words

$$|u_n-u|^{(0)}_{R^N}\to 0,n\to\infty.(2.5)$$

Further, (2.2) means in particular that for a fixed $k=1,2,...,N_1$ we have

$$\langle u_n-u_m\rangle^{(\alpha_k)}_{x_k,R^N}\to 0,n,m\to\infty.$$

And hence for a fixed $\varepsilon>0$ there exists $M_0>0$ such that

$$\langle u_n-u_m\rangle^{(\alpha_k)}_{x_k,R^N}\le\varepsilon,n,m>M_0$$

which can be reformulated exactly as

$$\sup_{x\in R^N,h>0}\frac{\left|(u_n-u_m)(x_1,...,x_k+h,...,x_M)-(u_n-u_m)(x)\right|}{h^{\alpha_k}}\le\varepsilon,\quad(2.6)$$

$$n,m>M_0.$$

Going, on the ground of (2.5), to the limit in the last inequality under any fixed $x\in R^N$ and $h>0$ as $m\to\infty$, we obtain

$$\sup_{x\in R^N,h>0}\frac{\left|(u_n-u)(x_1,...,x_k+h,...,x_M)-(u_n-u)(x)\right|}{h^{\alpha_k}}\le\varepsilon,n>M_0$$

that is

$$\langle u_n-u\rangle^{(\alpha_k)}_{x_k,R^N}\le\varepsilon,n>M_0.(2.7)$$

At last, from (2.1) it follows that for any fixed $k=1,2,...,N_1$ we have for all $n=1,2,...$

$$\sup_{x\in R^N}|u_n(x)|+\sup_{x\in R^N,h>0}\frac{\left|u_n(x_1,...,x_k+h,...,x_M)-u_n(x)\right|}{h^{\alpha_k}}\le C_1.$$

Fix $x\in R^N$ and $h>0$ in the last inequality and go to the limit as $n\to\infty$ on the ground of (2.5) to obtain after taking sup

$$\sup_{x\in R^N}|u(x)|+\sup_{x\in R^N,h>0}\frac{\left|u(x_1,...,x_k+h,...,x_M)-u(x)\right|}{h^{\alpha_k}}\le C_1$$

that is, since $k=1,2,...,N_1$ is arbitrary,

$$\|u\|_{C^{\alpha,0}_{x^{(1)}}(R^N)}\equiv|u|^{(\alpha,0)}_{x^{(1)},R^N}\le N_1C_1.(2.8)$$

Overall, (2.8) means that

$$u(x)\in C^{\alpha,0}_{x^{(1)}}(R^N)(2.9)$$

and (2.5) together with (2.7) mean that

$$\|u_n-u\|_{C^{\alpha,0}_{x^{(1)}}(R^N)}\to 0,n\to\infty.\quad(2.10)$$

Thus, on the ground (2.9) and (2.10) we conclude that $C^{\alpha,0}_{x^{(1)}}(R^N)$ is a Banach space.

The proof of the space $H^{\alpha,0}_{x^{(1)},p}(R^N)$ is a Banach space

Before going further let us stress the following circumstance

Remark 2. *Analogously to the $L_p(R^N)$ spaces, the elements $u(x)$ of the space $H^{\alpha,0}_{x^{(1)},p}(R^N)$ are in fact equivalence classes of functions. The functions of the same class are different from each other by a finite function $\varphi(x^{(2)})$ which depends only on variables $(x_{N_1+1},...,x_{N_1+N_2})$ and has zero $L_p(R^N)$-norm over whole space. This is possible only if $\varphi(x^{(2)})$ is different from zero only on a set $E\subset R^{N_2}$ of measure zero in R^{N_2}. In this case the set $R^{N_1}\times E\subset R^N$ also has measure zero (for example, as a countable union of sets with zero measure). Thus, the presence of such a function does not affect any integral functional (including an integral norm) or any Hölder seminorm with respect to variables $x^{(1)}$.*

Consequently, in our further reasonings we can consider any representative (function) $u(x)$ of the space $H^{\alpha,0}_{x^{(1)},p}(R^N)$ as a corrected one, if any, by adding some mentioned above function $\varphi(x^{(2)})$.

Thus, assume we are given a Cauchy sequence $\{u_n(x);n=1,2,...\}\subset H^{\alpha,0}_{x^{(1)},p}(R^N)$ in the space $H^{\alpha,0}_{x^{(1)},p}(R^N)$ with the properties

$$\|u_n\|_{H^{\alpha,0}_{x^{(1)},p}(R^N)}=\|u_n\|_{L_p(R^N)}+\sum_{k=1}^{N_1}\langle u_n\rangle^{(\alpha_k)}_{x_k,R^N}\le C_1,n=1,2,..,(3.1)$$

$$\|u_n-u_m\|_{H^{\alpha,0}_{x^{(1)},p}(R^N)}\to 0,n,m\to\infty,n,m=1,2,...(3.2)$$

that is

$$\|u_n-u_m\|_{L_p(R^N)}+\sum_{k=1}^{N_1}\langle u_n-u_m\rangle^{(\alpha_k)}_{x_k,R^N}\to 0,\quad(3.3)$$

$$n,m\to\infty,n,m=1,2,..$$

First of all, from (3.3) it follows that the sequence $\{u_n(x)\}$ is fundamental in $L_p(R^N)$ and hence it converges in $L_p(R^N)$ to some function $u(x)\in L_p(R^N)$,

$$\|u_n(x)-u(x)\|_{L_p(R^N)}\to 0,n\to\infty,n=1,2,..(3.4)$$

Our goal now will be to show that in fact we can choose such a representative for $u(x)\in L_p(R^N)$ that $u(x)\in H^{\alpha,0}_{x^{(1)},p}$ and

$$\|u_n(x)-u(x)\|_{H^{\alpha,0}_{x^{(1)},p}(R^N)}\to 0,n\to\infty,n=1,2,..(3.5)$$

Define the functions

$$V_n(x^{(2)})=\int_{R^{N_1}}\left|u_n(x^{(1)},x^{(2)})-u(x^{(1)},x^{(2)})\right|^p dx^{(1)}\ge 0.(3.6)$$

From (3.1), (3.4) and from the Fubini–Tonelli theorem it follows that these functions are well-defined almost everywhere in R^{N_2} and

$$\int_{R^{N_2}}|V_n(x^{(2)})|dx^{(2)}\to 0,n\to\infty,n=1,2,.$$

Moving on to a subsequence, if necessary, we can assume that $V_n(x^{(2)})$ converges to zero almost everywhere on R^{N_2}. This means that for almost all $x^{(2)}\in R^{N_2}$

$$\int_{R^{N_1}}\left|u_n(x^{(1)},x^{(2)})-u(x^{(1)},x^{(2)})\right|^p dx^{(1)}\to 0,n\to\infty.(3.7)$$

From (3.1), (3.7) it follows that for almost all $x^{(2)}\in R^{N_2}$

$$\|u_n(\cdot,x^{(2)})\|_{L_p(R^{N_1})}+\|u(\cdot,x^{(2)})\|_{L_p(R^{N_1})}\le C(x^{(2)}),n=1,2,.(3.8)$$

$$\|u_n(\cdot,x^{(2)})-u(\cdot,x^{(2)})\|_{L_p(R^{N_1})},n\to\infty.(3.9)$$

At the same time, relations (3.1), (3.3) give for all $x^{(2)}\in R^{N_2}$

$$\sum_{k=1}^{N_1}\langle u_n(\cdot,x^{(2)})\rangle^{(\alpha_k)}_{x_k,R^{N_1}}\le C_1,\sum_{k=1}^{N_1}\langle u_n(\cdot,x^{(2)})-u_m(\cdot,x^{(2)})\rangle^{(\alpha_k)}_{x_k,R^{N_1}}\to 0,(3.10)$$

$$n,m=1,2,...,$$

where the Hölder seminorm is taken with respect to all variables $x^{(1)}$ under any fixed $x^{(2)}\in R^{N_2}$. That is for all $x^{(2)}\in R^{N_2}\setminus E$ (the E is some set of measure zero, $|E|=0$) and for all functions $u_n(\cdot,x^{(2)})$ we have

$$\|u_n(\cdot,x^{(2)})\|_{L_p(R^{N_1})}+\sum_{k=1}^{N_1}\langle u_n(\cdot,x^{(2)})\rangle^{(\alpha_k)}_{x_k,R^{N_1}}\le C(x^{(2)}),n=1,2,.(3.11)$$

From (1.6) it follows that the norm (3.11) majorizes the usual norm in the Hölder space $C^{\alpha}(R^{N_1})$,

$$|u_n(\cdot,x^{(2)})|^{(0)}_{R^{N_1}}+\sum_{k=1}^{N_1}\langle u_n(\cdot,x^{(2)})\rangle^{(\alpha_k)}_{x_k,R^{N_1}}\le\quad(3.12)$$

$$\le C\left(\|u_n(\cdot,x^{(2)})\|_{L_p(R^{N_1})}+\sum_{k=1}^{N_1}\langle u_n(\cdot,x^{(2)})\rangle^{(\alpha_k)}_{x_k,R^{N_1}}\right)\quad(3.13)$$

with some absolute constant C. But this fact together with (3.9) and the second relation in (3.10) means that for $x^{(2)}\in R^{N_2}\setminus E$ the sequence $\{u_n(\cdot,x^{(2)})\}$ is fundamental in the Hölder space $C^{\alpha}(R^{N_1})$ with the norm (3.12) and hence it converges to $u(\cdot,x^{(2)})$ in this space. That is, in particular, for $x^{(2)}\in R^{N_2}\setminus E$ sequence converges to $u(\cdot,x^{(2)})$ uniformly and, in view of (3.10),

$$\sum_{k=1}^{N_1}\langle u(\cdot,x^{(2)})\rangle^{(\alpha_k)}_{x_k,R^{N_1}}\le C_1,x^{(2)}\in R^{N_2}\setminus E,(3.14)$$

where the constant C_1 does not depend on $x^{(2)}\in R^{N_2}\setminus E$.

Turn now to the remaining set of measure zero $E\subset R^{N_2}$. To cope with this set we choose an appropriate representative for $u_n(x)\subset H^{\alpha,0}_{x^{(1)},p}(R^N)$ - see Remark-2. This is done by modifying functions $u_n(x)$ on the set $E\times R^{N_1}$ in the following way

$$u_n(x)=u_n(x^{(1)},x^{(2)})\to\left[u_n(x^{(1)},x^{(2)})-u_n(0,x^{(2)})\right].$$

This is equivalent to adding to $u_n(x)$ the functions

$$\varphi(x^{(2)})=\begin{cases}-u_n(0,x^{(2)}),x^{(2)}\in E,\\ 0,x^{(2)}\in R^{N_2}\setminus E.\end{cases}$$

Such choice of representatives permits us to assume that for $x^{(2)}\in E$

$$u_n(0,x^{(2)})=0,x^{(2)}\in E,n=1,2,....(3.15)$$

This condition together with the Hölder property in (3.1) gives us that for $x^{(2)}\in E$ functions $u_n(x^{(1)},x^{(2)})$ satisfy (we denote $\gamma=max_k\{\alpha_k\}$)

$$|u_n(x^{(1)},x^{(2)})|\le C(C_1,\alpha,N_1)(1+|x^{(1)}|^{\gamma}).(3.16)$$

And analogously, for differences of functions, on the ground of (3.3),

$$|u_n(x^{(1)},x^{(2)})-u_m(x^{(1)},x^{|2|}|\le$$

$$\le C\left(\sum_{k=1}^{N_1}\langle u_n-u_m\rangle^{(\alpha_k)}_{x_k,R^N}\right)(1+|x^{(1)}|^{\gamma})\to 0,n,m\to\infty.(3.17)$$

That is for each $x^{(2)} \in E$ the sequence $\left\{u_n\left(x^{(1)}, x^{(2)}\right)\right\}$ is fundamental in $C^{\alpha}(K)$ for any compact set of R^{N_1} - in view of (3.3) and (3.17). Consequently, for each $x^{(2)} \in E$ and for any compact set $K \subset R^{N_1}$ the sequence $\left\{u_n\left(x^{(1)}, x^{(2)}\right)\right\}$ converges in the space $C^{\alpha}(K)$ to some function which we denote by the above symbol $u(x)=u\left(x^{(1)}, x^{(2)}\right)$. What is important for us at this point, the first, that this definition does not affect the $L_p\left(R^N\right)$-norm of $u(x)$ in (3.4). And the second, after moving on to the limit on compact sets at the fixed $x^{(2)} \in E$, the function $u(x)$ inherits the value of the Hölder seminorm of the functions $\left\{u_n\left(x^{(1)}, x^{(2)}\right)\right\}$ in (3.1) with respect to $x^{(1)}$, that is

$$\sum_{k=1}^{N_1} \left\langle u \right\rangle^{(\alpha_k)}_{x_k, R^{N_1}} \leq C_1 . \quad (3.18)$$

Overall, collecting now together the facts from (3.4), (3.14), and (3.18), we arrive at the assertion that

$$u(x) \in H^{\alpha,0}_{x^{(1)},p}\left(R^N\right), \left\|u_n(x)-u(x)\right\|_{H^{\alpha,0}_{x^{(1)},p}\left(R^N\right)} \to 0, n \to \infty.$$

And this finishes the proof of the fact that the space $H^{\alpha,0}_{x^{(1)},p}\left(R^N\right)$ is a Banach space.

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On functional spaces with partial Hölder property

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JEL classification: C10, C50, C60, C61, C80, C87, C90

We consider two types of spaces with Hölder property with respect to a part of independent variables. Besides Hölder seminorm one of them has norm which includes the uniform norm and the other has an integral norm instead of the uniform. As it was shown earlier, when the Hölder property holds with respect to all independent variables the uniform norm can be estimated over the integral norm and the Hölder seminorm. Thus, in the case of a bounded domain these two spaces coincide. But in the case of partial Hölder property the uniform norm cannot be estimated in that way and corresponding functions may be unbounded. For both of these spaces we prove that they are Banach spaces that is full normed spaces.

Keywords: Hölder property, partial Hölder property, functional space, integral norm, Banach space.

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